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The aim of this note is to study a discrete group quivariant shape theory by associating a projective system of equivriant CW complexes.

1. Introduction

Let G be a discrete group and X a G-space. For a subgroup H of G we denote $X^H = \{ x \in X;\ gx = x \text{ for every } g \in H \}$.

THEOREM 1. There is a functor C_G from the category of G-spaces and G-homotopy classes of G-maps into the pro-category of the subcategory consisting of G-CW complexes so that $C_G(X)$ has the universal property for equivariant shape theory with a system G-map $p: X \to C_G(X)$.

When G is a finite group, we know that a G-ANR has the G-homotopy type of a G-CW complex and vice versa. Also any numerable covering has a refinement of numerable G-equivariant covering. So we have

THEOREM 2. Let G be a finite group and X a G-space.

(1)The equivariant ANR shape of X is equivalent to $C_G(X)$.

(2)The natural system map p: $X \to C_G(X)$ is a shape equivalence. Moreover, if X is a normal G-space, then $p^H: X^H \to C_G(X)^H$ is a shape equivalence for every subgroup H of G.

(3) Let X and Y be normal G-spaces. Then, a G-map f: $X \to Y$ induces an equivalence $C_G(f)$: $C_G(X) \to C_G(Y)$ if and only if f^H: $X^H \to Y^H$ is a shape equivalence for every subgroup H of G.

The case when G is not a discrete group will be discussed elsewhere. Recently [16] treated the equivariant ANR shape for compact groups. The method of Seymour[11] which I mentioned in my talk is useful but not enouph to define an equivariant shape theory.

2. A quick review of shape theory

The general references are [2], [4] and [8]. Borsuk developed the theory of shape in 1968 when he was 60 years old. Actually Borsuk(1968) defined the shape for compact metric spaces, Mardešić-Segal (1971) for compact Hausdorff spaces, Fox(1972) for metric spaces, and Mardešić(1973) and K. Morita(1975) for topological spaces.

The most famous example is the shape of Warsaw circle WC = $\{(0,y);\ -2 \leqq y \leqq 1\} \cup \{(x, \sin(1/x));\ 0 < x \leqq 1/2\pi\} \cup \{(1/2\pi, y);\ -2 \leqq y \leqq 0\} \cup \{(x, -2);\ 0 \leqq x \leqq 1/2\pi\} \subset R^2$. Contracting {(0, y);

$-1 \leqq y \leqq 1\} \cup \{(x, \sin(1/x);\ 0 < x \leqq \varepsilon\}$ to a point, we get a continuous map $f\colon WC \to S^1$. Although $\pi_i(WC) = 0$ for any $i \geqq 0$, f seems a kind of "homotopy equivalence" in the following senses.

(1) We have systems of decreasing neighborhoods M_i of WC and N_i of S^1 in R^2 with $\cap M_i = WC$ and $\cap N_i = S^1$ such that there are homeomorphisms $f_i\colon M_i \to N_i$ with $f_i | M_{i+1} = f_{i+1}$. Fox's shape is a generalization of this.

(2) There are systems of coverings $\mathfrak{U}_i$ of WC and $\mathcal{V}_i$ of S^1 cofinal to all the coverings with $\mathfrak{U}_{i+1} < \mathfrak{U}_i$ and $\mathcal{V}_{i+1} < \mathcal{V}_i$ such that there are compatible isomorphisms of their nerves $N(\mathfrak{U}_i) \to N(\mathcal{V}_i)$. This implies the isomorphism of Čech (co)homology and is generalized to the shape in Morita's sense.

(3) $f^*\colon [S^1, K] \to [WC, K]$ is an isomorphism for any ANR or any CW complex K. The shape equivalence in Mardešić's sense is defined by this condition.

Now we give an exact definition of Morita's shape or Čech system.

DEFINITION. $(\Lambda, \leqq)$ is an inductive set if (1) $\lambda \leqq \lambda$, (2) $\lambda \leqq \mu$, $\mu \leqq \nu \Rightarrow \lambda \leqq \nu$ and (3) for any $\lambda, \lambda' \in \Lambda$ there is a $\mu \in \Lambda$ such that $\lambda \leqq \mu$ and $\lambda' \leqq \mu$.

DEFINITION. $(\{X_\lambda\}, \{p_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda\ (\lambda \leqq \lambda')\})$ is called a projective system if (1) $p_{\lambda\lambda} \simeq \mathrm{id}$ and (2) $\lambda \leqq \lambda' \leqq \lambda'' \Rightarrow p_{\lambda\lambda'} p_{\lambda'\lambda''} \simeq p_{\lambda\lambda''}$.

DEFINITION. $\{f_\mu\}\colon (\{X_\lambda\}, \{p_{\lambda\lambda'}\}, \Lambda) \to (\{Y_\mu\}, \{q_{\mu\mu'}\}, M)$ is called a system map if there are $\theta\colon M \to \Lambda$ and $f_\mu\colon X_{\theta(\mu)} \to Y_\mu$ such that $q_{\mu\mu'}f_{\mu'}p_{\theta(\mu')\lambda} \simeq f_{\mu}p_{\theta(\mu)\lambda}$ for $\mu \leqq \mu'$, $\theta(\mu') \leqq \lambda$ and $\theta(\mu) \leqq \lambda$.

For a space X we associate a projective system of CW complexes $C(X) = (\{X_\lambda\}, \{p_{\lambda\lambda'}\}, \Lambda)$ by

$\Lambda = \{\mathfrak{U}_\lambda$; all numerable coverings of X$\}$ and $\lambda' \geqq \lambda$ iff $\mathfrak{U}_{\lambda'} < \mathfrak{U}_\lambda$,

$X_\lambda = N(\mathfrak{U}_\lambda)$ and $p_{\lambda\lambda'}: N(\mathfrak{U}_{\lambda'}) \to N(\mathfrak{U}_\lambda)$,

where $p_{\lambda\lambda'}$ is defined by choosing $\tilde{p} = \tilde{p}_{\lambda\lambda'}$ so that $U_\alpha^{\lambda'} \subset U_{\tilde{p}(\alpha)}^{\lambda}$. The homotopy class of $p_{\lambda\lambda'}$ is independent of the choice of $\tilde{p}$. Moreover, the isomorphism class of C(X) is well defined and called the shape of X. Here, a pointwise finite covering $\mathfrak{U} = \{U_\alpha\}$ of X is called numerable if it admits a locally finite partition of unity $\{f_\alpha\}$ i.e., a family of continuous functions $f_\alpha: X \to [0, 1]$ with $\Sigma f_\alpha = 1$ and $f_\alpha^{-1}(0, 1] \subset U_\alpha$ such that $\{f_\alpha^{-1}(0,1]\}$ is a locally finite covering of X. By the locally finite partition of unity $\{f_\alpha\}$ subordinate to $\mathfrak{U}_\lambda$ we have a map $p_\lambda: X \to X_\lambda$ defined by $p_\lambda(x) = \Sigma f_\alpha(x)\langle U_\alpha\rangle$ where $\langle U_\alpha\rangle$ is the vertex corresponding to U_α. A different choice of the locally finite partition of unity gives another map contiguous to p_λ. So, the homotopy class of p_λ depends only on $\mathfrak{U}_\lambda$.

The shape associating a projective system $(\{X_\lambda\}, \{p_{\lambda\lambda'}\})$ of the subcategory $\mathfrak{W}$ is characterized by the following universal properties due to Mardešić:

(1) For any map $f: X \to K$ with $K \in \mathfrak{W}$ there exist a map $f_\lambda: X_\lambda \to K$ such that $f \simeq f_\lambda p_\lambda$, and

(2) if $f_\lambda p_\lambda \simeq g_\lambda p_\lambda$ then there is a $\lambda' \geqq \lambda$ such that $f_\lambda p_{\lambda\lambda'} \simeq g_\lambda p_{\lambda\lambda'}$.

So, with a system map $\{p_\lambda\}: X \to C(X)$ any CW shape is equivalent to C(X). Since ANR is homotopy equivalent to a CW complex and vice versa (Cf. [4,Appendix]), any ANR shape is also equivalent to C(X).

3. Equivariant shape $C_G(X)$ (Proof of Theorem 1)

Let G be a discrete group and X a G-space. A numerable covering $\mathfrak{U}=\{U_\alpha\}$ of X is called a numerable G-equivariant covering if $U_\alpha \in \mathfrak{U}$ implies $gU_\alpha = U_{g\alpha} \in \mathfrak{U}$ and $f_{g\alpha}(x) = f_\alpha(g^{-1}x)$ for any element g of G such that the following three sets have finite differences:

$\{g\in G;\ g\alpha=\alpha$ i.e., $f_{g\alpha} = f_\alpha\} \subset \{g\in G;\ gU_\alpha = U_\alpha\} \subset \{g\in G;\ gU_\alpha \cap U_\alpha \neq \phi\}$.

The nerves of the numerable G-equivariant coverings $\mathfrak{U}_\lambda$ of X induce a projective system $C_G(X)=\{C_{G,\lambda}(X)\}$ with a system G-map $\{p_\lambda: X \to C_{G,\lambda}(X)\}$ such that $p_\lambda \simeq_G p_{\lambda,\lambda'} p_{\lambda'}$. The G-homotopy classes of p_λ and $p_{\lambda\lambda'}$ are also well-defined by the argument using contiguity as in the non-equivariant case.

LEMMA 3.1. The natural system G-map $p: K \to C_G(K)$ is an equivalence as projective systems of G-CW complexes and G-homotopy classes of G-maps.

LEMMA 3.2.(Universal property for equivariant shape)

(1)For any G-CW complex K and any G-map $f: X \to K$ there exist a λ and a G-map $f_\lambda: C_{G,\lambda}(X) \to K$ such that $f \simeq_G f_\lambda p_\lambda$.

(2)If $f \simeq_G g_\lambda p_\lambda$ for any other G-map g_λ, then there is a $\nu \geqq \lambda$ such that $f_\lambda p_{\lambda\nu} \simeq_G g_\lambda p_{\lambda\nu}$.

Proof of Lemma 3.2 and Theorem 1. Lemma 3.2 is a detailed restatement of Theorem 1. Lemma 3.1 implies Lemma 3.2 in a standard way. In fact, there are a μ and a G-map $q:C_{G,\mu}(K) \to K$ such that q is a G-homotopy inverse to p_μ. With the system G-map $C_G(f): C_G(X) \to C_G(K)$ we have a λ and a G-map $f'_\lambda:C_{G,\lambda}(X) \to C_{G,\mu}(K)$. Now it suffices to define $f_\lambda = qf'_\lambda$. Hereafter we assume that any system map will be given by a level preserving morphism of systems equivalent to the original one. To prove (2) we note that $qC_{G,\mu}(f_\lambda)p^\lambda_\mu \simeq_G f_\lambda$ because q is a G-homotopy inverse to p^K_μ where we denote p_μ for $C_{G,\lambda}(X)$ by p^λ_μ. Since there is a G-homotopy inverse q' to p^λ_μ for some $\mu' \geqq \mu$ by Lemma 3.1, we have $f_\lambda q'C_{G,\mu'}(p_\lambda) \simeq_G qp^K_{\mu\mu'}C_{G,\mu'}(f)$. We also have $g_\lambda q''C_{G,\mu''}(p_\lambda) \simeq_G qp^K_{\mu\mu''}C_{G,\mu''}(f)$. Taking a ν with $\nu \geqq \mu'$ and $\nu \geqq \mu''$, we see that $f_\lambda q'C_{G,\mu'}(p_\lambda)p_{\mu'\nu} \simeq_G g_\lambda q''C_{G,\mu''}(p_\lambda)p_{\mu''\nu}$. But $q'C_{G,\mu'}(p_\lambda)p_{\mu'\nu} \simeq_G p_{\lambda\nu} \simeq_G q''C_{G,\mu''}(p_\lambda)p_{\mu''\nu}$, because q' and q'' are G-homotopy inverse to $p^\lambda_{\mu'}$ and $p^\lambda_{\mu''}$ respectively. q.e.d.

Proof of Lemma 3.1. We consider a G-map $\rho:|S(K)| \to K$ for the geometric realization of the singular complex of K. Since $|S(K)|^H = |S(K^H)|$, we see that ρ is a G-homotopy equivalence. Since a G-homotopy equivalence induces an equivalence $C_G(\cdot)$, the proof reduces to the following two lemmas.

LEMMA 3.3. For a G-space X, $|S(X)|$ admits a G-equivariant triangulation.

LEMMA 3.4. For a G-equivariantly triangulated G-space K, $p: K \to C_G(K)$ is an equivalence as projective systems of G-CW complexes and G-homotopy classes of G-maps.

Proof of Lemma 3.3. We know that there is a G-homeomorphism between $|S(X)|$ and $|SdS(X)|$ where $SdS(X)$ is a barycentric subdivision of the singular s.s. complex $S(X)$ of X. Note that the natural quotient map $q: |SdS(X)| \to |SdS(X)/G|$ restricts to a homeomorphism on any cell of $|SdS(X)|$. So, a triangulation of the regular CW complex $|SdS(X)/G|$ lifts to a G-equivariant triangulation of $|SdS(X)|$. q.e.d.

Proof of Lemma 3.4. For each vertex v we take an open star neighborhood U_v. Then, $v_1, \ldots, v_n$ are the vertices of the same simplex if and only if $U_{v_1} \cap \ldots \cap U_{v_n}$ is not empty. If necessary by taking a barycentric subdivision we may assume that if gv and v are in the same simplex of K then $gv = v$ and hence that $U_{gv} \cap U_v \neq \phi$ implies $gv = v$. So, $\mathfrak{U} = \{U_v\}$ is a pointwise finite covering. If we take $\tilde{f}_v(x)$ = the coefficient of v, the G-map $\tilde{p}: K \to N(\mathfrak{U})$ defined by $\{\tilde{f}_v\}$ is not only a bijection but also a G-homeomorphism. Note here that $\tilde{f}_v(gx) = \tilde{f}_v(x)$ if $gv = v$. Moreover, $\tilde{p}$ is G-homotopic to p defined by a locally finite G-equivariant partition unity by the argument using contiguity. This completes the proof of Lemma 3.4 and also that of Lemma 3.1 and Theorem 1. q.e.d.

4. The case when G is a finite group

Let G be a finite group and X a G-space. Then, (1) of Theorem 2 is a consequence of Theorem 1 and the fact that G-ANR has the

G-homotopy type of a G-CW complex and a G-CW complex is a G-ANR(Cf.[9] and [4,Appendix] or [17]). Pop[17] also treats the equivariant shape theory for a finite group G. (2) and (3) of Theorem 2 strengthen his result in the case that X is normal.

LEMMA 4.1. Let $G = \{g_1=e,\dots,g_n\}$ be a finite group. For any numerable covering $\mathfrak{U} = \{U_\alpha, f_\alpha\}$ of a G-space X we have a numerable G-equivariant covering $\mathcal{V}$ of X such that $\mathcal{V} < \mathfrak{U}$.

Proof. It suffices to take the covering $\mathcal{V}$ consisting of $g_1^{-1}U_{\alpha_1} \cap \dots \cap g_n^{-1}U_{\alpha_n}$ with $f_{\alpha_1}(g_1x)\cdots f_{\alpha_n}(g_nx)$. In fact, $g_i(g_1^{-1}U_{\alpha_1} \cap \dots \cap g_n^{-1}U_{\alpha_n}) \subset U_{\alpha_i}$ and the sum is $(\Sigma f_{\alpha_1})\cdots(\Sigma f_{\alpha_n}) = 1$. Note that we do not require $gV_\beta \cap V_\beta \neq \phi$ implies $gV_\beta = V_\beta$ for the numerable G-equivariant covering. q.e.d.

Proof of (2) of Theorem 2. Lemma 4.1 implies that $p: X \to C_G(X)$ is also a non-equivariant shape equivalence. Assume that X is a normal space. For a subgroup H of G any numerable covering $\mathfrak{U}_H$ of the closed subspace X^H extends to a numerable covering $\mathfrak{U}$ of X i.e., $\mathfrak{U}_H = \{U \cap X^H ; U \in \mathfrak{U}\}$. We may assume that if $U \cap X^H = \phi$ then U is not H-invariant for $U \in \mathfrak{U}$. So, we see that $C_G(X)^H = C_{W(H)}(X^H)$ for a normal G-space X where $W(H) = N(H)/H$ and $N(H) = \{ g \in G;\ gHg^{-1} = H\}$. Now we have proved (2) of Theorem 2 by considering X^H is a W(H)-space. q.e.d.

LEMMA 4.2. Let G be a finite group. Let X and Y be G-CW complexes and $h_H: X^H \to Y^H$ maps with $g_* h_H \simeq h_{H'} g_*$ for every pair of subgroups $H' \subset gHg^{-1}$ where $g_*(x) = gx$. Then, there is a G-map $f: X \to Y$ such that $f|X^H \simeq h_H$ for every subgroup H of G.

Proof. Choose a family of representatives $\{H_1, \ldots, H_n\}$ of conjugacy classes of subgroups of G. For G-0-cell $\Delta^0 \times G/H_i$ we define $f|X^0$ by $f(v \times gH_i/H_i) = g_* h_{H_i}(v)$. Assume that a G-map $f|X^{n-1}$ is defined and for $H = H_i$ there are given homotopies between $f|\sigma(\Delta^k \times H/H)$ and $h_H|\sigma(\Delta^k \times H/H)$ in Y^H which extend the homotopies on the boundaries as an induction hypothesis for $k < n$. Then, for a G-n-cell $\sigma: \Delta^n \times G/H \to X$ with $H = H_i$, $h_H|\sigma(\partial\Delta^n \times H/H)$ is homotopic to $f|\sigma(\partial\Delta^n \times H/H)$. We can now define $f|\sigma(\Delta^n \times H/H)$ by the homotopy on the collar and by h_H on the interior. Extending f on $\sigma(\Delta^n \times G/H)$ so that f becomes G-equivariant, $f|X^n$ satisfies also the induction hypothesis. So, we get a G-map $f: X \to Y$ such that $f|X^H \simeq h_H$.

q.e.d

Proof of (3) of Theorem 2. If $f: X \to Y$ induces an equivalence $C_G(f): C_G(X) \to C_G(Y)$, then $C_G(f)^H: C_G(X)^H \to C_G(Y)^H$ are equivalences. This means that $f^H: X^H \to Y^H$ are shape equivalences by (2) of Theorem 2. Now suppose that $f^H: X^H \to Y^H$ are shape equivalences. Then, also by (2) of Theorem 2 $C_G(f)^H: C_G(X)^H \to C_G(Y)^H$ are equivalences, that is, there are $\mu(H,\lambda) \geqq \lambda$ and G-maps $q_H: C_{G,\mu(H,\lambda)}(Y)^H \to C_{G,\lambda}(X)^H$ such that $q_H C_{G,\mu(H,\lambda)}(f)^H \simeq p^{X,H}_{\lambda,\mu(H,\lambda)}$ and $C_{G,\lambda}(f)^H q_H \simeq p^{Y,H}_{\lambda,\mu(H,\lambda)}$. By taking $\mu(\lambda) \geqq \mu(H,\lambda)$ for every H, we may assume that $\mu(H,\lambda) = \mu(\lambda)$ for any H. Note that if $H' \subset gHg^{-1}$ then we have the following diagram:

$$\begin{array}{ccccccccc}
C_{G,\mu''}(Y)^H & \xrightarrow{q_H} & C_{G,\mu'}(X)^H & \to & C_{G,\mu'}(Y)^H & \xrightarrow{q_H} & C_{G,\lambda}(X)^H & \to & C_{G,\lambda}(Y)^H \\
\downarrow g_* & & \downarrow g_* & & \downarrow g_* & & \downarrow g_* & & \downarrow g_* \\
C_{G,\mu''}(Y)^{H'} & \xrightarrow{q_{H'}} & C_{G,\mu'}(X)^{H'} & \to & C_{G,\mu'}(Y)^{H'} & \xrightarrow{q_{H'}} & C_{G,\lambda}(X)^{H'} & \to & C_{G,\lambda}(Y)^{H'}
\end{array}$$

Here, we denote $\mu''=(\mu(\lambda))$ and $\mu'=\mu(\lambda)$. Not necessarily $g_*q_H \simeq q_{H'}g_*$ but $g_*q_HC_{G,\mu'}(f)^Hq_H \simeq q_{H'}C_{G,\mu'}(f)^{H'}q_{H'}g_*$. This means that we may assume that $g_*q_H \simeq q_{H'}g_*$ for every H,H' and g by retaking $\mu(\lambda)$ big enouph. By Lemma 4.2 we get a new G-map $q:C_{G,\mu(\lambda)}(Y) \to C_{G,\lambda}(X)$ such that $q^H \simeq q_H$ for every subgroup H of G. Note that $q_HC_{G,\mu(\lambda)}(f)^H \simeq p^{X,H}_{\lambda,\mu(\lambda)}$ for every H. So, applying the same argument of Lemma 4.2, we can get a G-homotopy between $qC_{G,\mu(\lambda)}(f)$ and $p^X_{\lambda,\mu(\lambda)}$ and also a G-homotopy between $C_{G,\mu(\lambda)}(f)q$ and $p^Y_{\lambda,\mu(\lambda)}$. q.e.d.

References

[1] K. Borsuk: Theory of shape, Monografie Matematyczne 59, Polish Scientific Publishers, Warszawa, 1975.

[2] J. Dydak and J. Segal: Shape theory, An introduction, Lecture Notes in Math. 688, Springer, Berlin, 1978.

[3] H. Hastings and S. Waner: On Brown's construction for compact Lie group actions, Preprint.

[4] S. Mardešić and J. Segal: Shape theory, The inverse system approach, North-Holland Math. Library 26, Amsterdam, 1982.

[5] T. Matumoto: G-CW complexes and a theorem of J.H.C.

Whitehead, J. Fac. Sci. Univ. Tokyo, IA 18(1971),109-125.

[6] ---------: A complement to the theory of G-CW complexes, Japan. J. Math. 10(1984),353-374.

[7] T. Matumoto, N. Minami and M. Sugawara: On the set of free homotopy classes and Brown's construction, Hiroshima J. Math. 14(1984),359-369.

[8] K. Morita: shapeの理論，数学 28(1976),335-347.

[9] M. Murayama: On G-ANR's and their G-homotopy types, Osaka J. Math. 20(1983),479-512.

[10] G.B. Segal: Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34(1968),105-112.

[11] R.M. Seymour: Some functorial constructions on G-spaces, Bull. London Math. Soc. 15(1983),353-359.

[12] Yu.M. Smirnov: Shape theory and continuous transformation groups, Uspekhi Mat. Nauk 34:6(1979),119-123 = Russian Math. Surveys 34:6(1979),138-143.

[13] --------: Shape theory of G-pairs, Uspekhi Mat. Nauk 40:2 (1985),151-165 = Russian Math. Surveys 40:2(1985),185-203.

[14] S. Waner: Equivariant homotopy theory and Milnor's theorem, Trans. Amer. Math. 258(1980),351-368.

[15] J.H.C. Whitehead: On C^1-complexes, Ann. Math. 41(1940), 809-824.

[16] S.A. Antonian and S.Mardešić: Equivariant shape, Preprint.

[17] I. Pop: An equivariant shape theory, An. Ştinţ. Univ. "Al.I.Cuza" Iaşi s. Ia Mat. 30-2(1984),53-67.